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Fixed Point Theorem in Fuzzy Metric Space

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ABSTRACT
In this present paper on fixed point theorems in fuzzy metric space . we extended to Fuzzy Metric space generalisation of main theorem .
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I. INTRODUCTION

Now a day's some contractive condition is a central area of research on Fixed point theorems in fuzzy metric spaces satisfying. Zadeh[10] in 1965 was introduced fuzzy sets. After this developed and a series of research were done by several Mathematicians. Kramosil and Michlek [5] Helpern [4] in 1981 introduced the concept of fuzzy metric space in 1975 and fixed point theorems for fuzzy metric space. Later in 1994, A.George and P.Veeramani [3] modified the notion of fuzzy metric space with the help of t-norm. Fuzzy metric space, here we adopt the notion that, the distance between objects is fuzzy, the objects themselves may be fuzzy or not.

In this present papers Gahler [1],[2] investigated the properties of 2-metric space, and investigated contraction mappings in 2-metric spaces. We know that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in the Eucledian space, The idea of fuzzy 2-metric space was used by Sushil Sharma [8] and obtained some fruitful results. prove some common fixed point theorem in fuzzy -metric space by employing the notion of reciprocal continuity, of which we can widen the scope of many interesting fixed point theorems in fuzzy metric space.

II. PRELIMINARY NOTES

Definition 2.1. A tiangular norm $*$ (shortly t− norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d ∈ [0, 1] the following conditions are satisfied:

1. $a * 1 = a$;
2. $a * b = b * a$;
3. $a * b \le c * d$ whenever $a \le c$ and $b \le d$
4. $a * (b * c) = (a * b) * c$.

Example 2.2. Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all x, y ∈ X and t > 0, $M(x, y, t) = \dfrac{t}{t + d(x, y)}$. Then (X, M, $*$) is a fuzzy metric space and this metric d is the standard fuzzy metric.

Definition 2.3. A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said
(i).To converge to x in X if and only if $M(x_n, x, t) = 1$ for each $t > 0$.
(ii). Cauchy sequence if and only if $M(x_{n+p}, x_n, t) = 1$ for each $p > 0$, $t > 0$.
(iii**).**to be complete if and only if every Cauchy sequence in X is convergent in X.

Definition 2.4. A pair (f, g) or (A,S) of self maps of a fuzzy metric space (X, M, $*$) is said
(i). To be reciprocal continuous if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$ whenever there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x ∈ X.

(ii). semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x ∈ X.

Definition 2.5. Two self maps A and B of a fuzzy metric space (X, M, ∗) are said to be weak compatible if they commute at their coincidence points, that is Ax = Bx implies ABx = BAx.

Definition 2.6. A pair (A, S) of self maps of a fuzzy metric space (X, M, ∗) is said to be **Definition 2.7.** A binary operation ∗ : [0, 1] × [0, 1] × [0, 1] → [0, 1] is called a continuous t-norm if ([0, 1]), ∗) is an abelian topological monoid with unit 1 such that $a_1*b_1*c_1 \leq a_2*b_2*c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0, 1].

Definition 2.8. A sequence $\{x_n\}$ in a fuzzy 2-metric space (X, M, ∗) is said

(i),To converge to x in X if and only if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all a ∈ X and t > 0.

(ii).Cauchy sequence, if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ for all a ∈ X and p > 0, t > 0.

(iii). To be complete if and only if every Cauchy sequence in X is convergent in X.

Theorem 3.1 – Let A, B, S, T, L and M be a complete ε-chainable fuzzy metric space (X, M *) with continuous t-norm satisfying the conditions.

(1) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$

(2) AB = BA, ST = TS, LB = BL, MT = TM;
;

(3) there exists k ∈ (0, 1) such that

$$\min\{ M(Lx, My, kt)\{M(ABx, My, 2t), M(ABx, STy, t), M(ABx, Lx, t), M(STy, My, t)\} \geq 0$$

For every x, y ∈ X, α ∈ (0, 2) and t > 0. If L, AB is reciprocally continuous, semi-compatible maps. Then A, B, S, T, L and M have a unique common fixed point in X.

Proof : Let $x_0 \in$ X then from (1) there exists x_1, $x_2 \in$ X such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. In general we can find a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1}\ y = x_{2n+1}$ for t > 0 and α = 1 – q with q ∈ (0, 1) in (4), we have

$$M(y_{2n+1}, y_{2n+2}, kt) = M(Lx_{2n+2}, Mx_{2n+1}, kt)$$

$$\min\begin{Bmatrix} M(ABx_{2n+2}, Mx_{2n+1}, (2-(1-q)t)), M(ABx_{2n+2}, STx_{2n+1}, t) \\ M(ABx_{2n+2}, Lx_{2n+2}, t), M(STx_{2n+1}, Mx_{2n+1}, t) \end{Bmatrix} \geq 0$$

$$\min\begin{Bmatrix} M(y_{2n+1}, y_{2n+1}, ((1+q)t)), M(y_{2n+1}, y_{2n}, t) \\ M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t) \end{Bmatrix} \geq 0$$

$$= \min\{1, M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}$$

$$M(y_{2n+1}, y_{2n+2}, kt) \geq \min\ M(y_{2n+1}, y_{2n}, t)$$

Again x = x_{2n+2} and y = x_{2n+3} with α = 1-q with q ∈ (0, 1) in (4), we have

$$M(y_{2n+2}, y_{2n+3}, kt) = M(Lxy_{2n+2}, Mx_{2n+3}, kt)$$

$$\min\begin{Bmatrix} M(ABx_{2n+2}, Mx_{2n+3}, (1+q)t), M(ABx_{2n+2}, STx_{2n+3}, t) \\ M(ABx_{2n+2}, Lx_{2n+2}, t), M(STx_{2n+3}, Mx_{2n+3}, t) \end{Bmatrix} \geq 0$$

$$\min\begin{Bmatrix} M(y_{2n+1}, y_{2n+3}, ((1+q)t)), M(y_{2n+1}, y_{2n+2}, t) \\ M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, t) \end{Bmatrix} \geq 0$$

$$= \min\begin{Bmatrix} M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, qt), M(y_{2n+1}, y_{2n+2}, t), \\ M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, t) \end{Bmatrix} \geq 0$$

As t-norm continuous, letting q → 1 we have,

$$M\left(y_{2n+2}, y_{2n+3}, kt\right) \geq \min\left\{M\left(y_{2n+1}, y_{2n+2}, t\right), M\left(y_{2n+1}, y_{2n+3}, t\right)\right\}$$

Hence,

$$M\left(y_{2n+1}, y_{2n+3}, kt\right) \geq M\left(y_{2n+1}, y_{2n+2}, t\right)$$

Therefore for all n; we have

$$M\left(y_n, y_{n+1}, t\right) \geq M\left(y_n, y_{n-1}, t/k\right) \geq M\left(y_n, y_{n-1}, t/k^2\right) \geq \ldots \geq M\left(y_n, y_{n-1}, t/k^n\right) \to 1 \text{ as } n \to \infty.$$

For any $t > 0$. For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \,\varepsilon\, N$ such that $M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for all $n > n_0$. For m, n ε N, we suppose $m \geq n$. Then we have that

$$M\left(y_n, y_m, t\right) \geq M\left(y_n, y_{n+1}, t/m-n\right), M\left(y_{n+1}, y_{n+2}, t, m-n\right), * \ldots$$

$$* M\left(y_{m-1}, y_m, t/m-n\right) > (1-\in) * (1-\in) * (1-\in) * \ldots * (1-\in) \geq (1-\in)$$

Hence $\{y_n\}$ is a Cauchy sequence in X; that is $y_n \to z$ in X; so its subsequences Lx_{2n}, STx_{2n+1}, ABx_{2n}, Mx_{2n+1} also converges to z. Since X is ε-chainable, there exists ε-chain from x_n to x_{n+1}, that is, there exists a finite sequence $x_n = y_1, y_2, \ldots y_t = x_{n+1}$ such that $M(y_i, y_{i-1}, t) > 1-\varepsilon$ for all $t > 0$ and i = 1, 2, ...l. Thus we have $M(x_n, x_{n+1}, t) > M(y_1, y_2, t/l) * M(y_2, y_3, t/l) * \ldots * M(y_{i-1}, y_i, t/l) > (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) * \ldots * (1 - \varepsilon) * \geq (1 - \varepsilon)$, and so $\{x_n\}$ is a Cauchy sequence in X and hence there exists $z \in X$ such that $x_n \to z$. Since the pair of (L, AB) is reciprocal continuous; we have $\lim_{n\to\infty} L(AB)x_{2n} \to Lz$ and $\lim_{n\to\infty} AB(L)x_{2n} \to ABz$ and the semi compatibility of (L, AB) which gives $\lim_{n\to\infty} AB(L)x_{2n} \to ABz$, therefore Lz = ABz. We claim

Lz = ABz = z.

Step 1 : Putting x = z and $y = x_{2n+1}$ with α = 1 in (4), we have

$$M\left(Lz, Mx_{2n+1}, kt\right) \geq \min\left\{\begin{array}{l} M\left(ABz, Mx_{2n+1}, t\right), M\left(ABz, STx_{2n+1}, t\right), \\ \qquad M\left(ABz, Lz, t\right), M\left(STx_{2n+1}, Mx_{2n+1}, t\right) \end{array}\right\}$$

Letting n → ∞; we have

$$M\left(Lz, z, kt\right) \geq \min\left\{M\left(Lz, z, t\right), M\left(Lz, z, t\right), M\left(Lz, Lz, t\right), M\left(z, z, t\right)\right\}$$

i.e.

z = Lz = ABz.

Step 2 : Putting x = Bz, $y = x_{2n+1}$ with α = 1 in (4), we have

$$M\left(L(Bz), Mx_{2n+1}, kt\right) \geq \min\left\{\begin{array}{l} M\left(AB(Bz), Mx_{2n+1}, t\right), M\left(AB(Bz), STx_{2n+1}, t\right) \\ \qquad M\left(AB(Bz), L(Bz), t\right), M\left(STx_{2n+1}, Mx_{2n+1}, t\right) \end{array}\right\}$$

Since LB = BL, AB = BA, so L(Bz) = B(Lz) = Bz and AB(Bz) = B(ABz) = Bz letting n → ∞; we have

$$M\left(Bz, z, kt\right) \geq \min\left\{M\left(Bz, z, t\right), M\left(Bz, z, t\right), M\left(Bz, z, t\right), M\left(z, z, t\right)\right\}$$

i.e.

$$M\left(bz, z, kt\right) \geq M\left(Bz, z, t\right)$$

Therefore

$L(X) \subseteq ST(X)$, there exists $u \in X$, such that z = Lz = Stu. Putting $x = x_{2n}$, y = u with α = 1 in (4), we have

$$M\left(Lx_{2n}, Mu, kt\right), \min\left\{\begin{array}{l} M\left(ABx_{2n}, Mu, 2t\right), M\left(ABx_{2n}, STu, t\right), \\ \qquad M\left(ABx_{2n}, Lx_{2n}, t\right), M\left(STu, Mu, t\right) \end{array}\right\} \geq 0$$

Letting n → ∞; we have

$$M\left(z, Mu, kt\right), \min\left\{M\left(z, Mu, t\right), M\left(z, z, t\right), M\left(z, z, t\right), M\left(z, Mu, t\right)\right\} \geq 0$$

i.e.

$$M\left(z, Mu, kt\right) \geq M\left(z, Mu, t\right)$$

Therefore

Z = Mu = STu.

Since M is ST-absorbing; then

$$M\left(STu, STMu, kt\right) \geq M\left(STu, Mu, t/R\right) = 1$$

i.e. STu = STMu => z = STz.

Step 4 : Putting $x = x_{2n}$, y = z with α = 1 in (4), we have

$$M\left(Lx_{2n}, Mz, kt\right), \min\left\{\begin{matrix} M\left(ABx_{2n}, Mz, t\right), M\left(ABx_{2n}, STz, t\right), \\ M\left(ABx_{2n}, Lx_{2n}, t\right), M\left(STz, Mz, t\right)\end{matrix}\right\} \geq 0$$

Letting n → ∞; we have

$$M\left(z, Mz, kt\right), \min\left\{M\left(z, Mz, t\right), M\left(z, z, t\right), M\left(z, z, t\right), M\left(z, Mz, t\right)\right\} \geq 0$$

i.e.

$$M\left(z, Mz, kt\right) \geq M\left(z, Mz, t\right)$$

Therefore

z = Mz = STz.

Step 5 : Putting x = x_{2n}, y = Tz with α = 1 in (4), we have

$$M\left(Lx_{2n}, M\left(Tz\right), kt\right), \min\left\{\begin{matrix} M\left(ABx_{2n}, M\left(Tz\right), t\right), M\left(ABx_{2n}, ST\left(Tz\right), t\right), \\ M\left(ABx_{2n}, Lx_{2n}, t\right), M\left(ST\left(Tz\right), M\left(Tz\right), t\right)\end{matrix}\right\} \geq 0$$

Since MT = TM, ST = TS therefore M(Tz) = T(Mz) = Tz, ST(Tz) = T(STz) = Tz;
Letting n → ∞; we have

$$M\left(z, Tz, kt\right), \min\left\{M\left(z, Tz, t\right), M\left(z, z, t\right), M\left(z, z, t\right), M\left(Tz, Tz, t\right)\right\} \geq 0$$

i.e.

$$M\left(z, Tz, kt\right) \geq M\left(z, Tz, t\right)$$

Therefore

z = Tz = Sz = Mz.

Hence

z = Az = Bz = Lz = Sz = Mz = Tz.

Uniqueness : Let w be another fixed point of A, B, L, S, M and T. Then putting x = u, y = w with α = 1 in (4) we have

$$M\left(Lu, Mw, kt\right), \min\left\{\begin{matrix} M\left(ABu, Mw, t\right), M\left(ABu, STw, t\right), \\ M\left(ABu, Lu, t\right), M\left(STw, Mw, t\right)\end{matrix}\right\} \geq 0$$

$$\min\left\{M\left(u, w, t\right), M\left(u, w, t\right), M\left(u, u, t\right), M\left(w, w, t\right)\right\} \geq 0$$

Therefore

$$M\left(u, w, kt\right) \geq M\left(u, w, t\right)$$

Hence

z = w.

Corollary 3.2 : Let A, B, S, T, L and M be a complete ε-chainable fuzzy metric space (X,M,*) with continuous t-norm satisfying the conditions (1) to (3) of theorem 3.1 and ; (5) there exists $k \in \left(0,1\right)$ such that

$$\min\{ M\left(Lx, My, kt\right), \left\{\begin{matrix} M\left(ABx, My, 2t\right), M\left(ABx, STy, t\right), \\ M\left(ABx, Lx, t\right), M\left(STy, My, t\right), M\left(STy, Lx, 2t\right)\end{matrix}\right\}\} \geq 0$$

For every x, y ∈ X, α ε (0, 2) and t > 0. If L, AB is reciprocally continuous, semi-compatible maps. Then A, b, S, T, L and M have a unique common fixed point in X.

REFERENCES

[1]. Y.J. Cho, B.K. Sharma, D.R. Sahu Semi-compatibility and fixed point Math. Apon. 42(1995), 1, 91-98.

[2]. S.H. Cho, J.H. Jung, On common fixed point theorems in fuzzy metric spaces, int. mathematical Forum, 1, 2006, 29, 144`-1451.

[3]. M.Erceg, Metric spaces in fuzzy set theory, J.Math. Anal. Appl., 69 (1979), 205-230.

[4]. M.Grabiec fixed points in fuzzy metric space fuzzy sets and system, 27, (1998) 385-389.

[5]. A.George and P.Veermani., On some results in fuzzy metric spaces. Fuzzy sets and sysem 64 (1994), 395-399.

[6]. G.Jungck, compatible mappings and fixed points, Internat. J.Math. Math. Sci. 9(4) (1986), 771-779.

[7]. O.Kramosil, J.Michelak, Fuzzy metric and statistical metric space, Kybernetika, 11 (1975), 326-334.

[8]. U.Mishra, A.S. Ranadive, & D.Gopal, Fixed point theorems via absorbing maps; Thai J.Math., vol. 6(1) (2008) 49-60.

[9]. A.S. Ranadive, A.P.Chouhan, Common fixed point theorem in fuzzy metric spaces, (Accepted) Proc. Of Math. Soc., B.H.U. Vol. 24.

[10]. A.S.Ranadive, D.Gopa, U.Mishra, On some open problems of common fixed point theorems for a pair of non-compatible self-maps. Proc. Of Math. Soc., B.H.U. Vol. 20 (2004), 135-141.

[11]. B.Singh, and S.Jain, semi-compatibility and fixed point theorem in fuzzy metric space using implicit relation Int. Jour of Math. And Math. Sci. 2005 : 16 (2005) 2617-2619.

[12]. B.Schwizer and Skalar, Statistical metric spaces, Pacific J.Math. 10 (1960), 313-334.

[13]. R.Vasuki, Common fixed point for r-weakly commuting maps in fuzzy metric spaces, Indian J.Pure and Applied Mathematice (1999), 419-423.

[14]. L.A. Zadeh, Fuzzy sets Inform. And Cont., 89 (1965) 338-353.